WEIGHTED NORM INEQUALITIES
FOR HOMOGENEOUS SINGULAR INTEGRALS

JAVIER DUOANDIKOETXEA

ABSTRACT. We prove weighted norm inequalities for homogeneous singular integrals when only a size condition is assumed on the restriction of the kernel to the unit sphere. The same results hold for the operator obtained by modifying the centered Hardy-Littlewood maximal operator over balls with a degree zero homogeneous function and also for the maximal singular integral.

1. INTRODUCTION

Given a function Ω over the unit sphere S^{n-1} of $\mathbf{R}^n$, we will consider a singular integral operator T_Ω given by

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy$$

and a related maximal operator

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y') f(x-y)|\, dy\,,$$

where $y' = y|y|^{-1}$. The Calderón-Zygmund method of rotations [CZ] proves that if $\Omega \in L^1(S^{n-1})$, then M_Ω is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and the same is true for T_Ω if Ω is odd. If Ω is even, then T_Ω is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, when $\Omega \in L\log^+ L(S^{n-1})$ (i.e., $|\Omega|\max(0, \log|\Omega|) \in L^1(S^{n-1})$) and $\int_{S^{n-1}} \Omega = 0$.

In this paper we are interested in weighted norm inequalities for T_Ω and M_Ω; that is, we look for locally integrable nonnegative functions w in $\mathbf{R}^n$ such that for some C independent of f

$$(1) \qquad \int |T_\Omega f|^p w \le C \int |f|^p w\,,$$

and the same for M_Ω. We denote by $L^p(w)$ the L^p space with respect to the measure $w(x)\, dx$ in $\mathbf{R}^n$ so that (1) is equivalent to saying that T_Ω is a bounded operator in $L^p(w)$. The class of weights w for which (1) holds for T_Ω (resp. M_Ω) will be denoted by $W_p(T_\Omega)$ (resp. $W_p(M_\Omega)$). Also, given q, the

class of weights w for which (1) holds simultaneously for all T_Ω (resp. M_Ω), $\Omega \in L^q(S^{n-1})$ is denoted by $W_p(T_q)$ (resp. $W_p(M_q)$).

Weighted inequalities like (1) for $w(x) = |x|^\alpha$ are in [MW]. More general weights were studied in [KW1] but only assuming a Dini-type condition on Ω. In this paper we show that the operator T_Ω is bounded in $L^p(w)$ for the weights w considered in [KW1] and without assuming a Dini-type condition on Ω. Also, we show that the same is true for M_Ω. The technique we use is based on the alternative approach to the unweighted case presented in [DR1], where weighted inequalities for bounded Ω were also obtained. These results are proved in §3, after some technical lemmas given in §2. In §4 we study radial weights which include, for both T_Ω and M_Ω, the weights of power type of [MW] for T_Ω. In §5 some weights depending on Ω are also determined. In §6 we consider the maximal singular integral associated to T_Ω in order to deduce the almost everywhere convergence of the truncated integrals defining T_Ω for functions in $L^p(w)$. In §7 we extend the results to other singular integrals.

We recall that A_p $(p > 1)$ is the class of weights w for which the Hardy-Littlewood maximal function (corresponding to $\Omega \equiv 1$ and denoted throughout this paper by M) is bounded in $L^p(w)$ and was characterized by Muckenhoupt as those locally integrable nonnegative functions such that for some finite constant C and any cube Q in $\mathbf{R}^n$

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} \le C.$$

A_1 is the class of weights w for which M satisfies a weak-type estimate in $L^1(w)$ and is characterized by $Mw(x) \le Cw(x)$ a.e. For most of the results related to weighted norm inequalities we will refer to the monograph [GR] rather than to the original papers.

Notice finally that the inequality

$$(2) \qquad M_\Omega f(x) \le C(Mf^{q'}(x))^{1/q'} \qquad (1/q + 1/q' = 1)$$

holds as a consequence of Hölder's inequality.

After this manuscript was ready the author learned that some of the results of this paper had been obtained independently by D. Watson (see [W]).

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2. SOME TECHNICAL LEMMAS

Given $\Omega \in L^q(S^{n-1})$, $q > 1$, we define

$$E_j f(x) = 2^{-jn}\int_{2^j < |y| \le 2^{j+1}} |\Omega(y')||f(x-y)|\,dy$$

and if, moreover, $\int_{S^{n-1}}\Omega = 0$, we also define

$$T_j f(x) = \int_{2^j < |y| \le 2^{j+1}} \frac{\Omega(y')}{|y|^n} f(x-y)\,dy.$$

Then $M_\Omega f(x) \sim \sup_j E_j(|f|)(x)$ (i.e., their quotient is bounded above and below by positive finite constants);

$$|T_j f(x)| \le C E_j(|f|)(x) \quad \text{and} \quad T_\Omega f(x) = \sum_{j=-\infty}^{\infty} T_j f(x)$$

for $f \in \mathscr{S}(\mathbf{R}^n)$. We also define

$$g_\Omega(f)(x) = \left(\sum_{j=-\infty}^{\infty} |T_j f(x)|^2 \right)^{1/2}$$

and denote by $W_p(g_\Omega)$ the L^p-weights for g_Ω.

Following [DR1] we make a new decomposition of T_Ω. Choose a real function $\psi \in \mathscr{C}^\infty(\mathbf{R}^n)$, supported in $\frac{1}{2} < |\xi| < 2$, and such that

$$\sum_{k=-\infty}^{\infty} |\psi(2^k \xi)|^2 = 1 \quad \forall \xi \neq 0.$$

Define S_k as $(S_k f)^\wedge(\xi) = \psi(2^k \xi) \hat{f}(\xi)$ so that $\sum_k S_k^2 = \mathrm{Id}$. Let $\tilde{T}_k f = \sum_{j=-\infty}^{\infty} T_j S_{j+k}^2 f$; then $T_\Omega = \sum_k \tilde{T}_k$.

In [DR1] we proved that given $1 < p < \infty$, there exists an $\alpha = \alpha(p) > 0$ such that

$$(3) \qquad \|\tilde{T}_k f\|_p \le C 2^{-\alpha|k|} \|f\|_p.$$

For $p = 2$ this inequality is given by a Fourier transform estimate. In [DR1] it was shown that for any $p_0 \neq 2$, (3) holds with a constant independent of k. Given p, take p_0 such that p is between p_0 and 2 and interpolate to get (3). Alternatively, one could prove that the $H^1 - L^1$ constant of $\tilde{T}_k$ grows as $C(1 + |k|)$ and apply interpolation again.

Lemma 1. *Let $w \in A_p$. If the vector-valued inequality*

$$(4) \qquad \left\| \left(\sum_j |E_j f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \le C \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(w)}$$

holds, then $w^\theta \in W_p(T_\Omega)$ for $0 \le \theta < 1$. Also, (4) holds if $w \in W_p(M_\Omega)$ and $p \le 2$.

Proof. Decompose T_Ω as before. Then we have the following chain of inequalities:

$$\|\tilde{T}_k f\|_{L^p(w)} \le C \left\| \left(\sum_j |T_j S_{j+k} f|^2 \right)^{1/2} \right\|_{L^p(w)}$$

$$\le C \left\| \left(\sum_j |S_{j+k} f|^2 \right)^{1/2} \right\|_{L^p(w)} \le C \|f\|_{L^p(w)},$$

where the first and third inequalities are deduced from the weighted Littlewood-Paley theory with $w \in A_p$ (see [K]), and the second inequality is a consequence of $|T_j f| \le E_j(|f|)$ and the hypothesis (4). Using Stein and Weiss' interpolation theorem with change of measure (see [BL, p. 115]) we interpolate with (3) and summing in k we get the first part of the lemma.

If $w \in W_p(M_\Omega)$, then the inequality

$$\left\| \left(\sum_j |M_\Omega f_j|^p \right)^{1/p} \right\|_{L^p(w)} \leq C \left\| \left(\sum_j |f_j|^p \right)^{1/p} \right\|_{L^p(w)}$$

is trivial. Moreover, since M_Ω is a positive operator,

$$\sup_j |M_\Omega f_j(x)| \leq M_\Omega \left(\sup_j |f_j| \right)(x)$$

pointwise and therefore also in the $L^p(w)$-norm. If $p < 2$, one can interpolate (see [GR, p. 481] for the vector-valued interpolation) to get (4) with M_Ω instead of E_j which is enough. □

Lemma 2. *If $w \in W_{p'}(T_\Omega)$, then $w^{1-p} \in W_p(T_\Omega)$.*

The lemma is easily proved by duality since T_Ω is (essentially) selfadjoint.

Given Ω, we define $\Omega_0(u) = |\Omega(u)| - \|\Omega\|_1/|S^{n-1}|$. Then Ω_0 is in the same L^q class as Ω and has mean value zero. The pointwise inequality

$$E_j|f|(x) \leq C T_j^0 |f|(x) + C M f(x)$$

holds, where T_j^0 is defined as T_j but with Ω_0 instead of Ω. Then

$$M_\Omega f(x) \leq C g_{\Omega_0}(|f|)(x) + C M f(x),$$

which proves the following lemma.

Lemma 3. *If $w \in A_p \cap W_p(g_{\Omega_0})$, then $w \in W_p(M_\Omega)$.*

We obtain weighted inequalities for the square function g_Ω by considering the linear operators $T_{\varepsilon,\Omega} f = \sum_j \varepsilon_j T_j f$, where $\varepsilon = \{\varepsilon_j\}$ is a sequence with $\varepsilon_j = +1$ or -1. If $T_{\varepsilon,\Omega}$ is bounded in $L^p(w)$ with a constant independent of ε, then the usual argument with Rademacher functions [S] gives the boundedness of g_Ω in $L^p(w)$. In practice, $T_{\varepsilon,\Omega}$ is decomposed as T_Ω and (3) still holds.

Lemma 4. *Let $q < 2$ and $p > 2$. If $M_{\Omega^{2-q}}$ is bounded in $L^{(p/2)'}(w^{1/(1-p/2)})$, then the vector-valued inequality (4) in Lemma 1 holds for p.*

Proof. Applying the Cauchy-Schwarz inequality we get

$$|E_j f_j(x)|^2 \leq C 2^{-jn} \int_{2^j < |x-y| \leq 2^{j+1}} |\Omega((x-y)')|^{2-q} |f_j(y)|^2 \, dy,$$

and, if $p > 2$, there exists u with unit norm in $L^{(p/2)'}(w)$ such that

$$\left\| \left(\sum_j |E_j f_j|^2 \right)^{1/2} \right\|_{L^p(w)}^2 = \int \sum_j |E_j f_j(x)|^2 u(x) w(x) \, dx.$$

Majorizing $|E_j f_j(x)|^2$ by the above bound and integrating first in x we have

$$\left\|\left(\sum_j |E_j f_j|^2\right)^{1/2}\right\|_{L^p(w)}^2 \leq C \int \sum_j |f_j(y)|^2 M_{\Omega^{2-q}}(uw)(y)\,dy$$

$$\leq C \left\|\left(\sum_j |f_j|^2\right)^{1/2}\right\|_{L^p(w)}^2 \left(\int |M_{\Omega^{2-q}}(uw)|^{(p/2)'} w^{1/(1-p/2)}\right)^{1/(p/2)'},$$

and it suffices to apply the hypothesis of the lemma to see that the last term is bounded. $\square$

3. Weights in Muckenhoupt classes

Theorem 5. (i) $A_{p/q'} \subset W_p(M_q) \cap W_p(T_q)$ if $q' \leq p < \infty$;

(ii) $w^{1-p} \in W_p(M_q) \cap W_p(T_q)$ if $w \in A_{p'/q'}$ and $1 < p \leq q$.

Proof. From (2) and the weighted inequalities for the Hardy-Littlewood maximal function one gets $A_{p/q'} \subset W_p(M_q)$ if $q' < p < \infty$. Since M_Ω is bounded in $L^p(w)$ $\forall w \in A_1$ for $p > q'$, and in L^p for $p < q'$, interpolation with change of measure (see [BL, p. 119]) gives $A_1 \subset W_{q'}(M_q)$. Here and in the sequel we use the basic fact that

$$(5) \qquad w \in A_p \Rightarrow w^{1+\varepsilon} \in A_p \quad \text{for some } \varepsilon > 0,$$

which implies $\{w^\theta/w \in A_p,\ 0 \leq \theta < 1\} = A_p$.

Let $q \geq 2$. Applying Lemma 1 we deduce $A_{p/q'} \subset W_p(T_q)$ in $q' \leq p \leq 2$ and for $p > 2$ we can use the extrapolation theorem of Rubio de Francia [R, Theorem 3, p. 539]. In this case, the part of (ii) corresponding to $W_p(T_q)$ follows from (i) and Lemma 2. To get (ii) for $W_p(M_q)$ we apply Lemma 3. In fact, the weighted inequalities for T_Ω are also valid for $T_{\varepsilon,\Omega}$, and duality can be used for these operators as in Lemma 2.

Assume now that $\frac{4}{3} \leq q < 2$. Then, if $\Omega \in L^q(S^{n-1})$, we have $\Omega^{2-q} \in L^{q/(2-q)}(S^{n-1})$ and $2 < q/(2-q)$ so that we can apply the preceding results to Ω^{2-q}. Due to the fact that $(q/(2-q))' = q'/2$, when $p \geq q'$ we have $w^{1/(1-p/2)} \in W_{(p/2)'}(M_{\Omega^{2-q}})$ if $w \in A_{p/q'}$, according to (ii). Then Lemma 4 applies together with Lemma 1 and (5) to give $A_{p/q'} \subset W_p(T_q)$. Reasoning as above, Lemmas 2 and 3 provide (ii).

Next we consider the values of q in the range $\frac{8}{7} \leq q < \frac{4}{3}$ for which $\frac{4}{3} \leq q/(2-q) < 2$. The results just proved are then valid for Ω^{2-q} and the same reasoning works. A bootstrapping argument is then applied to get the theorem for any $q > 1$. $\square$

We do not know whether the analogue of (5) is true for $W_p(M_\Omega)$, i.e.,

$$w \in W_p(M_\Omega) \Rightarrow w^{1+\varepsilon} \in W_p(M_\Omega) \quad \text{for some } \varepsilon > 0$$

(analogously for $W_p(M_q)$). Also a duality result like Lemma 2,

$$w \in W_{p'}(M_q) \Rightarrow w^{1-p} \in W_p(M_q),$$

would have avoided the use of the bootstrapping argument. Both properties together would give $W_p(T_q) = W_p(M_q)$.

More weights can be obtained by interpolating those in (i) and (ii).

4. RADIAL WEIGHTS

Theorem 6. M_Ω and T_Ω are bounded in $L^p(|x|^\alpha)$ if

$$\max\left(-n, -1 - (n-1)\frac{p}{q'}\right) < \alpha < \min\left(n(p-1), p-1 + (n-1)\frac{p}{q'}\right).$$

Proof. Assume $\alpha \leq 0$. When $p \geq q'$, the range $-n < \alpha \leq 0$ is given by Theorem 5 since those weights are in $A_1(\mathbf{R}^n)$; but they can also be obtained with the method of rotations. Take then $p < q'$. Denoting by M_u the Hardy-Littlewood maximal function in the direction $u \in S^{n-1}$ and applying Hölder's inequality we have

$$M_\Omega f(x) \leq \int_{S^{n-1}} |\Omega(u)| M_u f(x) \, d\sigma(u)$$

$$\leq \|\Omega\|_{L^q(S^{n-1})}^{1/p'} \left(\int_{S^{n-1}} |\Omega(u)|^{(1-q/p')p} |M_u f(x)|^p \, d\sigma(u)\right)^{1/p}.$$

Then

$$\int_{\mathbf{R}^n} |M_\Omega f(x)|^p w(x) \, dx$$

$$\leq C \int_{S^{n-1}} |\Omega(u)|^{(1-q/p')p} \int_{\mathbf{R}^n} |M_u f(x)|^p w(x) \, dx \, d\sigma(u)$$

$$\leq C \int_{S^{n-1}} |\Omega(u)|^{(1-q/p')p} \int_{\mathbf{R}^n} |f(x)|^p M_u w(x) \, dx \, d\sigma(u)$$

(see [GR, p. 150] for the last inequality) so that $w \in W_p(M_\Omega)$ whenever

$$\int_{S^{n-1}} |\Omega(u)|^{(1-q/p')p} M_u w(x) \, d\sigma(u) \leq C w(x) \quad \text{a.e.}$$

Applying Hölder's inequality again with exponents $q/(1-q/p')p = 1/(1 - p/q')$ and its conjugate q'/p, it is enough to prove

$$(6) \qquad \left(\int_{S^{n-1}} |M_u w(x)|^{q'/p} \, d\sigma(u)\right)^{p/q'} \leq C w(x) \quad \text{a.e.}$$

Take $w(x) = |x|^\alpha$ with $\alpha < 0$. Since w is radial, the left-hand side of (6) is also radial and we can assume $x = (|x|, 0, \ldots, 0)$. It is easy to compute $M_u w(x)$ because the restriction of w to the line through x with direction u is symmetric and decreasing with respect to the projection of the origin on this line. So, if θ is the angle of u with the x_1 axis, we get

$$M_u w(x) \sim C|x|^\alpha \quad \text{when } |\sin\theta| \geq |\cos\theta| \text{ or } -1 < \alpha \leq 0;$$

$$\sim C|x|^\alpha |\sin\theta|^{1+\alpha} \quad \text{when } |\sin\theta| \leq |\cos\theta| \text{ and } \alpha < -1;$$

$$\sim C|x|^\alpha \log\frac{1}{|\sin\theta|} \quad \text{when } |\sin\theta| \leq |\cos\theta| \text{ and } \alpha = -1,$$

and (6) holds whenever

$$\int_0^{\pi/4} (\sin\theta)^{(1+\alpha)q'/p} (\sin\theta)^{n-2} \, d\theta < +\infty,$$

which is true if $-1 - (n-1)p/q' < \alpha$, as the hypothesis of the theorem states.

For T_Ω, the values $-n < \alpha \le 0$ for $p \ge q'$ are still given by Theorem 5, and Lemma 1 applies when $1 < p \le \min(2, q')$. If $2 < q'$, then one applies interpolation to fill the gap $2 < p < q'$. The result for $\alpha > 0$ is given by duality (Lemma 2). To apply the duality to M_Ω we pass through Lemma 3 as above. $\square$

Theorem 6 was proved for T_Ω by Muckenhoupt and Wheeden [MW] in a different way and they also showed that the range is optimal. Their method also works for M_Ω. Our approach is based on the method of rotations and shows that any operator given as

$$(7) \qquad \int_{S^{n-1}} \Omega(u) R_u f(x) \, d\sigma(u),$$

where R_u denotes the directional operator defined from some one-dimensional operator R bounded in $L^p(v, \mathbf{R})$ for all $v \in A_1(\mathbf{R})$, is bounded in $L^p(w)$ for all w satisfying (6) with $(M_u w^s)^{1/s}$ (any $s > 1$) instead of $M_u w$. This modification comes from the fact that one has to use the inequality

$$\int |R_u f(x)|^p w(x) \, dx \le C \int |f(x)|^p (M_u w^s(x))^{1/s} \, dx$$

with $s > 1$ (since $(M_u w^s)^{1/s}$ is an A_1-weight) instead of the one used in the proof of the theorem with $s = 1$, which is false in general (for example if R is the Hilbert transform).

Recalling that $|x|^\alpha \in A_p(\mathbf{R}^n)$ if and only if $-n < \alpha < n(p-1)$, one can verify that there are more power weights in Theorem 6 than in Theorem 5.

Theorem 7. *Let* $w(x) = v_1(|x|)v_2^{1-p}(|x|)$, *where either* $v_i \in A_1(\mathbf{R}^+)$ *and is decreasing or* $v_i^2 \in A_1(\mathbf{R}^+)$, $i = 1, 2$. *Then* M_Ω *and* T_Ω *are bounded in* $L^p(w)$. *This result is valid for* M_Ω *when* $\Omega \in L^1(S^{n-1})$, *and also for* T_Ω *if moreover* Ω *is odd.*

Proof. For a radial function v such that $v(x) = v_0(|x|)$ it is shown in [CHS] that

$$M_u v(x) \le C(M v_0^{2+\delta}(|x|))^{1/(2+\delta)} \quad \forall \delta > 0$$

and, if v is decreasing,

$$M_u v(x) \le C \, M v_0(|x|),$$

where M is the Hardy-Littlewood maximal function on $\mathbf{R}^+$. With the assumptions of the theorem, $v_i(|x|)$, $i = 1, 2$, satisfies the A_1 condition for M_u uniformly in u so that $w(x) = v_1(|x|)v_2^{1-p}(|x|)$ satisfies the A_p condition uniformly. Then

$$(8) \qquad \int |M_u f(x)|^p w(x) \, dx \le C \int |f(x)|^p w(x) \, dx$$

with C independent of u.

The theorem is now immediate for M_Ω since

$$\|M_\Omega f\|_{L^p(w)} \le \int_{S^{n-1}} |\Omega(u)| \, \|M_u f\|_{L^p(w)} \, d\sigma(u).$$

Applying (8) with the Hilbert transform instead of M we get the theorem for T_Ω when Ω is odd. When Ω is even, the usual modification of the method of rotations (see [CZ]) does the job. $\square$

According to P. Jones' factorization theorem [GR, p.436] the weights $w(x) = w_0(|x|)$ such that $w_0^2 \in A_p(\mathbf{R}^+)$ are included in the above statement. For power weights $|x|^\alpha$ this theorem only gives $-1 < \alpha < p - 1$ which is the best possible result for $\Omega \in L^1(S^{n-1})$ (not included in Theorem 6).

Again the result can be generalized to all operators represented as in (7) provided R is bounded in $L^p(v, \mathbf{R})$ for all $v \in A_p(\mathbf{R})$.

Proceeding as in Theorem 7 one can generalize the results of [DR2]:

Theorem 8. *Let* $\{R_j\}_{j=1}^n$ *be the n Riesz transforms in $\mathbf{R}^n$, and w a radial weight of those appearing in Theorem 7. Then, for $1 < p < \infty$, there exists a constant C depending only on w_0 and p and not on n such that*

$$\left\| \left(\sum_{j=1}^n |R_j f|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

An analogous theorem exists for higher order Riesz transforms. All one needs is to implement into the proofs of [DR2] the weighted inequality (8) (with the Hilbert transform) and realize that the constant in it is independent of n.

Let us finally indicate that some radial weights are also obtained for singular integrals with variable kernel:

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x, y')}{|y|^n} f(x - y) \, dy,$$

where $\sup_x \|\Omega(x, \cdot)\|_{L^q(S^{n-1})} < +\infty$. Theorem 7 holds with $p \geq q'$ and Theorem 6 holds with $-n < \alpha \leq 0$ and $p \geq q'$ (a better range appears in [MW]). The details are left to the reader.

5. WEIGHTS DEPENDING ON Ω

The spirit of the extrapolation theorem is that the weighted L^2-inequalities, for example, contain all the information about the weighted L^p-inequalities. A weaker (and much easier) statement would be that the weighted L^2-inequalities contain the unweighted L^p-inequalities. Assuming $p > 2$, for instance, for some $u \in L^{(p/2)'}(\mathbf{R}^n)$ of unit norm we have

$$\|T_\Omega f\|_p^2 = \int |T_\Omega f|^2 u$$

and one only needs $w \in W_2(T_\Omega)$ such that $u \leq w$ a.e. and $\|w\|_{(p/2)'} \leq C\|u\|_{(p/2)'}$ in order to deduce the L^p boundedness of T_Ω in a standard way. When $\Omega \in L^q(S^{n-1})$ and $q \geq 2$, Theorem 5 assures that $A_1 \subset W_2(T_\Omega)$ and $w = (Mu^s)^{1/s}$ for any s between 1 and $(p/2)'$ does the job. But if $q < 2$, none of the above theorems provides enough weights to deduce the L^p-inequalities. However, we know that T_Ω is bounded in L^p, $1 < p < \infty$. The restriction on the amount of weights comes from the fact that we have been producing weights valid for all $\Omega \in L^q$, rather than for a particular one, and the purpose of the following theorem is to show how to construct weights in $W_2(T_\Omega)$ such that the L^p-inequalities follow.

Theorem 9. *Let* $\Omega \in L^q(S^{n-1})$, $q < 2$, $1 < s < r < \infty$, $u \in L^r(\mathbf{R}^n)$, *and* A *be the norm of the operator* $M_{\Omega^{2-q}} M$ *in* $L^{r/s}(\mathbf{R}^n)$. *Then*

$$w = \left(\sum_{n=0}^{\infty} \frac{1}{(2A)^n} (M_{\Omega^{2-q}} M)^n u^s \right)^{1/s} \in W_2(T_\Omega),$$

where $(M_{\Omega^{2-q}} M)^n$ *stands for the composition* n *times of* $M_{\Omega^{2-q}} M$.

Proof. We claim that inequality (4) in Lemma 1 holds when $p = 2$ for any function w satisfying $M_{\Omega^{2-q}} w \leq C w$ a.e. In fact, proceeding as in Lemma 4 (with $p = 2$, so that $u \equiv 1$),

$$\int \sum_j |E_j f_j|^2 w \leq C \int \sum_j |f_j|^2 M_{\Omega^{2-q}} w.$$

The series in the statement of the theorem converges in the norm of $L^{r/s}(\mathbf{R}^n)$ and since M and $M_{\Omega^{2-q}}$ are positive operators with $v \leq \min(Mv, M_{\Omega^{2-q}} v)$ we have

$$\max(Mw^s, M_{\Omega^{2-q}} w^s) \leq M_{\Omega^{2-q}} Mw^s \leq \sum_{n=0}^{\infty} \frac{1}{(2A)^n} (M_{\Omega^{2-q}} M)^{n+1} u^s \leq 2Aw^s$$

so that w^s is an A_1-weight (hence A_2), which satisfies $M_{\Omega^{2-q}} w^s \leq 2Aw^s$. By applying Lemma 1 with w^s in the inequality (4) we deduce that $w \in W_2(T_\Omega)$. $\square$

The construction of w in the theorem follows from the algorithm of Rubio de Francia in [R]. It is clear that it satisfies $u \leq w$ a.e. (just taking $n = 0$ in the series) and $\|w\|_r \leq C\|u\|_r$, the conditions required above to deduce the L^p-inequalities with $p > 2$.

6. The Maximal Singular Integral

We define the following maximal operator associated to the singular integral T_Ω:

$$T_\Omega^* f(x) = \sup_{\varepsilon \to 0} |T^\varepsilon f(x)| = \sup_{\varepsilon \to 0} \left| \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x - y) \, dy \right|.$$

It is well known that the boundedness of T_Ω^* in $L^p(w)$ implies the almost everywhere existence of $\lim_{\varepsilon \to 0} T^\varepsilon f(x)$, the principal value integral defining T_Ω for $f \in L^p(w)$. We apply again the method of [DR1] or the method of rotations in order to get weighted norm inequalities for T_Ω^*.

Theorem 10. (i) *Let* $\Omega \in L^q(S^{n-1})$, $q > 1$, *let* $1 < p < \infty$, *and let* w *be as in Theorem 5, 6, or 7. Then* T_Ω^* *is bounded in* $L^p(w)$.

(ii) *Let* $\Omega \in L^1(S^{n-1})$ *be odd. Then* T_Ω^* *is bounded in* $L^p(w)$ *for* w *as in Theorem 7.*

Proof. (i) Assume $2^k \leq \varepsilon < 2^{k+1}$. Then

$$|T^\varepsilon f(x)| \leq C\, M_\Omega f(x) + |T^{2^k} f(x)|,$$

so that it is enough to take dyadic values of ε. On the other hand,

$$|T^{2^k} f(x)| = \left| \sum_{j=k}^{\infty} T_j f \right| \leq C\, M(T_\Omega f) + C\, Mf + \left| (\delta - \Phi_k) * \sum_{j=k}^{\infty} T_j f \right|$$

is the inequality proved in [DR1] with δ the Dirac delta and $\Phi_k = 2^{-kn}\Phi(2^{-k}\cdot)$, where Φ is in the Schwartz class $\mathscr{S}(\mathbf{R}^n)$, $\widehat{\Phi}(0) = 1$, and $\operatorname{supp}\Phi \subset B(0, 1)$. Therefore, it is enough to consider the last term, which in turn is bounded by

$$\sup_k \left| (\delta - \Phi_k) * \sum_{j=k}^{\infty} T_j f \right| \leq \sum_{j=0}^{\infty} \sup_k |(\delta - \Phi_k) * T_{j+k} f|.$$

Each term in the sum is bounded by $\sup_k |(\delta + |\Phi|_k) * M_\Omega f|$ so that all the weights obtained for M_Ω are valid for it. Moreover, in any unweighted L^{p_0}-norm it has an exponential decay (use the Fourier transform in $p_0 = 2$ and interpolate for other values).

(ii) When Ω is odd we have

$$T^\varepsilon f(x) \leq \pi \int_{S^{n-1}} |\Omega(u)| |H_u^* f(x)| \, d\sigma(u),$$

where H^* is the maximal Hilbert transform. The right-hand side of the inequality is an operator of the form (7) so that the proof of Theorem 7 applies to it.

When Ω is even and belongs to $L^q(S^{n-1})$, $q > 1$, the method of rotations can be applied by using several operators of the type (7) because T_Ω has to be written as a combination of Riesz transforms applied to operators with odd kernels (see [CZ] for the details). The method used in (ii) is then applicable to these operators and we get the weights in Theorems 6 and 7. $\square$

7. SOME EXTENSIONS AND REMARKS

(a) Theorems 5 and 9 can be extended to the case where the kernel of the singular integral contains a radial bounded function h, i.e.,

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - y) \, dy.$$

These operators were studied by Fefferman in [F] by using interpolation of an analytic family of operators, but it was shown in [DR1] that the method sketched in §2 of this paper also works. Again this method gives Theorems 5 and 9 and the corresponding part of Theorem 10. However, since the method of rotations is not applicable here, we do not have the analogues of Theorems 6 and 7.

(b) Another modification concerns the use of a pseudonorm associated to a nonisotropic group of dilations in $\mathbf{R}^n$ instead of the euclidean norm, i.e.,

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\|y\| > \varepsilon} \frac{h(\|y\|)\Omega(y')}{\|y\|^a} f(x - y) \, dy,$$

where $\|\cdot\|$ denotes the pseudonorm, a is the homogeneous dimension of $\mathbf{R}^n$ with respect to this group, $h \in L^\infty(0, \infty)$, and $\Omega \in L^q(S^{n-1})$, $q > 1$. The definition of the A_p-class of weights must be modified according to the present structure of $\mathbf{R}^n$ in the standard way. Here, even if $h \equiv 1$, the method of rotations cannot be applied because the directional operators would correspond to maximal functions and Hilbert transforms along homogeneous curves, for

which the appropriate weighted inequalities are still unknown. So, we again get Theorems 5 and 9 and part of Theorem 10, and miss Theorems 6 and 7.

(c) Multiple singular integrals in product spaces and strong maximal operators can also be treated by the methods in this paper. By using products of two spaces, $\mathbf{R}^n \times \mathbf{R}^m$, in order to simplify notations, the operators are

$$T_\Omega f(x_1, x_2) = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_{\substack{|y_1| > \varepsilon_1 \\ |y_2| > \varepsilon_2}} \frac{\Omega(y_1', y_2')}{|y_1|^n |y_2|^m} f(x_1 - y_1, x_2 - y_2) \, dy_1 \, dy_2$$

and

$$M_\Omega f(x_1, x_2) = \sup_{r_1, r_2 > 0} \frac{1}{r_1^n r_2^m} \int_{\substack{|y_1| < r_1 \\ |y_2| < r_2}} |\Omega(y_1', y_2')| \, |f(x_1 - y_1, x_2 - y_2)| \, dy_1 \, dy_2.$$

Muckenhoupt's definition of A_p in §1 is now modified by taking averages over products of cubes (one gets the weights for the strong maximal function, see [GR]). Looking at [D], where the unweighted inequalities were obtained, the reader will easily find the way to modify Lemmas 1 and 4 and Theorems 5 and 9. As in the unweighted case we are not able to handle the maximal singular integral.

Theorems 6 and 7 are extended to M_Ω for any intregrable Ω and to T_Ω for Ω odd in both variables by using the method of rotations and biradial weights, i.e., $w(x) = |x_1|^{\alpha_1} |x_2|^{\alpha_2}$ with α_1 and α_2 in the appropriate range in Theorem 6 and $w(x) = w_1(|x_1|)w_2(|x_2|)$ with w_1, w_2 as in Theorem 7. For these weights T_Ω^* is also bounded. When Ω is not odd in both variables, the method of rotations seems difficult to apply even in the unweighted case.

(d) If we define $W_1(M_\Omega)$ as

$$W_1(M_\Omega) = \{w / M_\Omega w \le Cw \text{ a.e.}\},$$

then any weight $w \in W_p(M_\Omega)$ such that $w^{1-p'} \in W_{p'}(M_\Omega)$ (in particular, all those appearing in our theorems) can be factorized as $w = w_0 w_1^{1-p}$, $w_0, w_1 \in W_1(M_\Omega)$ (see [R]). If we were able to prove the duality property mentioned in §3, namely

$$w \in W_p(M_\Omega) \Leftrightarrow w^{1-p'} \in W_{p'}(M_\Omega),$$

then any weight in $W_p(M_\Omega)$ could be factorized as before.

(e) Weak-type $(1, 1)$ weighted inequalities have been studied for T_Ω in [KW1, KW2] and for T_Ω^* in [CS], in both cases by assuming a Dini-type condition on Ω. The situation is much more complicated when we merely assume a size condition on Ω, as is shown by the fact that even the unweighted weak-type estimate was obtained only thirty years after the L^p one was proved (see [C, CR, H1]). One could conjecture that the inequality

$$(9) \qquad\qquad w(\{x : |T_\Omega f(x)| > \lambda\}) \le \frac{C}{\lambda} \int |f| w$$

is satisfied when either $w(x) = |x|^\alpha$ with $-1 - (n-1)/q' < \alpha \le 0$ (from Theorem 6), or when $w(x) = w_0(|x|)$ and either $w_0 \in A_1(\mathbf{R}^+)$ and is decreasing or $w_0^2 \in A_1(\mathbf{R}^+)$ (from Theorem 7). Also, a similar conjecture could be made for M_Ω. In the case of power weights and $n = 2$, this result was proved by Hofmann [H2]. The conjecture from Theorem 5 is that (9) holds when $w^{q'} \in A_1(\mathbf{R}^n)$.

(f) When $\Omega \in L \log^+ L(S^{n-1})$, T_Ω is bounded in $L^p(w)$ for $w(x) = |x|^\alpha$, $-1 < \alpha < p - 1$, and also for w as in Theorem 7 by using the method of rotations. Also T_Ω is bounded in $L^2(w)$ for w as in Theorem 9 with $q = 1$. In this case, one can extrapolate from the result for $q > 1$ since the constant has a size of the order $\|\Omega\|_q (q - 1)^{-1}$.

REFERENCES

[BL]   J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.

[CZ]   A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[CS]   O. N. Capri and C. Segovia, *Behaviour of L^r-Dini singular integrals in weighted L^1-spaces*, Studia Math. **92** (1989), 21–36.

[CHS]   A. Carbery, E. Hernández, and F. Soria, *Estimates for the Kakeya maximal operator on radial functions in $\mathbb{R}^n$*, Harmonic Analysis, ICM-90 Satellite Conference Proceedings, S. Igari (ed.), Springer-Verlag, Tokyo, 1991, pp. 41–50.

[C]   M. Christ, *Weak type $(1, 1)$ bounds for rough operators*, Ann. of Math. (2) **128** (1988), 19–42.

[CR]   M. Christ and J. L. Rubio de Francia, *Weak type $(1, 1)$ bounds for rough operators. II*, Invent. Math. **93** (1988), 225–237.

[D]   J. Duoandikoetxea, *Multiple singular integrals and maximal functions along hypersurfaces*, Ann. Inst. Fourier (Grenoble) **36** (1986), 185–206.

[DR1]   J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.

[DR2]   _____, *Estimations indépendantes de la dimension pour les transformées de Riesz*, C. R. Acad. Sci. Paris **300** (1985), 193–196.

[F]   R. Fefferman, *A note on singular integrals*, Proc. Amer. Math. Soc. **74** (1979), 266–270.

[GR]   J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.

[H1]   S. Hofmann, *Weak $(1, 1)$ boundedness of singular integrals with nonsmooth kernel*, Proc. Amer. Math. Soc. **103** (1988), 260–264.

[H2]   _____, *Weighted weak-type $(1, 1)$ inequalities for rough operators*, Proc. Amer. Math. Soc. **107** (1989), 423–435.

[K]   D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. Amer. Math. Soc. **259** (1980), 235–254.

[KW1]   D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[KW2]   _____, *A note on singular integrals with weights*, Proc. Amer. Math. Soc. **81** (1981), 391–397.

[MW]   B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc. **161** (1971), 249–258.

[R]   J. L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. **106** (1984), 533–547.

[S]   E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

[W]   D. Watson, *Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. **60** (1990), 389–399.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA, 28049 MADRID, SPAIN

Current address: Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

E-mail address: mtpduzuj@lg.ehu.es